Mirror Design Issues

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1 Summary

To identify critical design issues with flexible mirror dynamics, especially those issues that are at the interface between mirror, structure, and control design, we are motivated by the following questions.

1. How many actuators should be used?
2. How thick must the mirror be?
3. How much wind can be tolerated?
4. How to design the control? Will passive spring supports work? Is force feedback possible? Is additional damping (passive or active) necessary?

These are not questions that can be readily answered by a numerical finite element model. It will require a number of iterations until we settle on the right concept.

To get early qualitative insight, we looked for the simplest model that could approximate the mirror dynamics to obtain some analytical formulas related to the above list of queries. The model used was that of a simply supported Euler-Bernoulli beam (see Fig. 1), for which we selected the

Figure 1: Simply supported mirror segment beam model

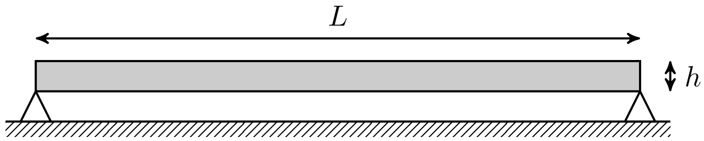
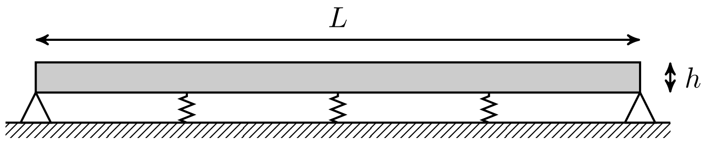


Figure 2: Simply supported mirror segment beam model with 3 regularly spaced springs



geometry and material properties to emulate a rectangular section of the Colossus mirror. As compared with a full three-dimensional model for the mirror, our beam model is simpler and has less complex dynamics, especially on what concerns the boundary conditions. In this sense, our results should be taken as cautiously optimistic.

Using this beam model we observed the following:

1. The mirror modal frequencies are

$$\omega_i = \sqrt{\frac{E\,b\,h^3}{12\rho}} \left(\frac{i\pi}{L}\right)^2, \qquad\qquad i = 1, \ldots, \infty, \tag{1}$$

   where h is the thickness of the mirror, $L = 8$ m is the diameter of the mirror, b is the width of the rectangular section we selected, 32 cm, $E = 50$ G Pa and $\rho = 2.5$ g/cm^3 for glass.

2. If we select $h = 2.5$ cm (as currently proposed for Colossus) then

$$\omega_1 \approx 0.8\,\mathrm{Hz} \tag{2}$$

   The corresponding "stiffness" of the mirror is also very small, of about 10^4 N/m, which could produce, according to the model of course, a displacement at the center of the mirror of the order of of centimeters in response to a 1 N distributed force applied to the mirror segment. This local stiffness is the one that should be used when designing local controllers as explained in an earlier note.

3. The mirror must have thickness $h > 1.5$ m thick to meet the required nanometer level of shape error.

4. Now suppose we add n springs of stiffness k to support the mirror. Then the modal frequencies of the new system (mirror plus support springs) can be expressed in terms of the mirror modal

frequencies ω_i as,

$$\omega_i(\kappa) = \sqrt{\frac{E\,b\,h^3}{12\rho}\left(\frac{i\pi}{L}\right)^4 + \frac{\kappa}{L\rho}}, \qquad i = 1, \ldots, \infty \tag{3}$$

where

$$\kappa = (1+n)\,k. \tag{4}$$

5. The number of springs n of stiffness k required to achieve a 10 nanometer static performance in response to a distributed force of 1 N with a $h = 2.5$ cm thick mirror must satisfy

$$\kappa = (1+n)\,k > 10^8\,\mathrm{N/m}$$

This number is essentially invariant with mirror thickness in the range 2.5 cm – 40 cm. The dynamic performance can be two orders of magnitude worse depending on the level damping of the mirror systems. The corresponding first mode of vibration is

$$\omega_1(\kappa) \approx 70\,\mathrm{Hz}. \tag{5}$$

6. Based on a covariance analysis for vibration from wind noise using only the first mode of the mirror vibration, the maximum allowable wind intensity W (variance) for a guaranteed mirror displacement error variance Q is given by

$$\frac{W}{Q} = \frac{\zeta L^2 \pi^2\, \rho^2 \omega_1(\kappa)^3}{4}, \tag{6}$$

where ζ is the damping ratio of the mirror. For $\zeta = 0.01$ and a guaranteed mirror displacement standard deviation of 10 nm, we obtain that a maximum wind standard deviation of 0.1 N is possible with $\kappa \approx 10^9$ N/m. For a maximum wind standard deviation of 10 N we need $\kappa \approx 5 \times 10^{11}$ N/m.

2 The Details

2.1 The Euler-Bernoulli Beam

The undamped dynamics of an Euler-Bernoulli beam simply-supported at $r = 0$ and $r = L$ are described by the ordinary differential equations

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \sum_j \psi_i(r_j)\, f_j(t) + \int_0^L \psi_i(r)\, g(r,t)\, dr, \qquad i = 1, \ldots, \infty \tag{7}$$

where the f_j's, τ_k's are impulsive forces and torques, g is a distributed force and

$$\omega_i = \sqrt{\frac{EI}{\rho}} \left(\frac{i\pi}{L}\right)^2 . \tag{8}$$

The functions ψ's are the mode shapes

$$\psi_i(r) = \sqrt{\frac{2}{\rho L}} \sin \frac{i\pi r}{L}, \tag{9}$$

and the beam's transversal displacement is given by

$$\mu(r,t) = \sum_{i}^{\infty} \psi_i(r)\, q_i(t). \tag{10}$$

2.1.1 Euler-Bernoulli Beam on Springs

Let us now add $j = 1, \dots, n$ springs of stiffness k at the symmetric locations

$$r_j = \frac{jL}{n+1}, \qquad j = 1, \dots, n. \tag{11}$$

Each spring produces a force

$$f_j(t) = -k\, \mu(r_j, t) = -k \sum_{\ell=1}^{\infty} \psi_\ell(r_j)\, q_\ell(t), \qquad j = 1, \dots, n. \tag{12}$$

The total spring force to be applied at each mode is equal to

$$\sum_{j=1}^{n} \psi_i(r_j)\, f_j(t) = k \sum_{j=1}^{n} \psi_i(r_j) \sum_{\ell=1}^{\infty} \psi_\ell(r_j)\, q_\ell = k \sum_{\ell=1}^{\infty} \sum_{j=1}^{n} \psi_\ell(r_j) \psi_i(r_j)\, q_\ell \qquad i = 1, \dots, \infty \tag{13}$$

Note that

$$\sum_{j=1}^{n} \psi_i(r_j)^2 = \frac{1}{2L\rho} \left(1 + 2n - \frac{\sin\left(\frac{(1+2n)\pi}{1+n} i\right)}{\sin\left(\frac{\pi}{1+n} i\right)} \right) = \frac{1+n}{L\rho} \tag{14}$$

because

$$\sin\left(\frac{(1+2n)\pi}{1+n} i\right) = \sin\left(\frac{[2(1+n)-1]\pi}{1+n} i\right) = \sin\left(2i\pi - \frac{\pi}{1+n} i\right) = -\sin\left(\frac{\pi}{1+n} i\right) \tag{15}$$

and

$$\sum_{j=1}^{n} \psi_\ell(r_j)\psi_i(r_j) = \frac{1}{2L\rho} \left(\frac{\sin\left(\frac{(1+2n)\pi}{2(1+n)}(i-\ell)\right)}{\sin\left(\frac{\pi}{2(1+n)}(i-\ell)\right)} - \frac{\sin\left(\frac{(1+2n)\pi}{2(1+n)}(i+\ell)\right)}{\sin\left(\frac{\pi}{2(1+n)}(i+\ell)\right)} \right) = 0, \quad i \neq \ell \tag{16}$$

because

$$\sin\left(\frac{(1+2n)\pi}{2(1+n)}(i\pm\ell)\right)=\sin\left(\frac{[2(1+n)-1]\pi}{(2(1+n)}(i\pm\ell)\right)=$$
$$\sin\left((i\pm\ell)\pi-\frac{\pi}{1+n}(i\pm\ell)\right)=-\varsigma(i\pm\ell)\sin\left(\frac{\pi}{1+n}i\right) \quad (17)$$

where $\varsigma(i\pm\ell)=-1$ if $i\pm\ell$ is odd and $\varsigma(i\pm\ell)=1$ if $i\pm\ell$ is even. The conclusion is that

$$\sum_{j=1}^{n}\psi_i(r_j)\,f_j(t)=k\sum_{\ell=1}^{\infty}\sum_{j=1}^{n}\psi_\ell(r_j)\psi_i(r_j)\,q_\ell=\frac{(1+n)k}{L\rho}\,q_i(t) \qquad i=1,\ldots,\infty \quad (18)$$

Substitution in the dynamic equations leads to

$$\ddot{q}_i(t)+\omega_i(\kappa)^2q_i(t)=\sum_j\psi_i(r_j)\,f_j(t)+\int_0^L\psi_i(r)\,g(r,t)\,dr, \qquad i=1,\ldots,\infty \quad (19)$$

where

$$\omega_i(\kappa)=\sqrt{\omega_i^2+\frac{(1+n)k}{L\rho}}=\sqrt{\frac{EI}{\rho}\left(\frac{i\pi}{L}\right)^4+\frac{\kappa}{L\rho}} \quad (20)$$

are the new modal frequencies in terms of the stiffness term

$$\kappa:=(1+n)\,k. \quad (21)$$

2.1.2 Distributed Time-varying Force

Application of a uniformly distributed time-varying force amounts to having

$$g(r,t)=\frac{F(t)}{L} \quad (22)$$

with which

$$\int_0^L\psi_i(r)\,g(r,t)\,dr=\frac{F(t)}{i\pi}\sqrt{\frac{2}{L\rho}}\,(1-\cos(i\pi))\,, \qquad i=1,\ldots,\infty \quad (23)$$

Note that $\cos(i\pi)=-1$ if i is even and $\cos(i\pi)=1$ if i is odd, leading to equations of motion of the form

$$\ddot{q}_i(t)+\omega_i(\kappa)^2q_i(t)=\frac{2F(t)}{i\pi}\sqrt{\frac{2}{L\rho}}, \qquad i=1,3,\ldots,\infty, \quad (24)$$
$$\ddot{q}_i(t)+\omega_i(\kappa)^2q_i(t)=0, \qquad i=2,4,\ldots,\infty \quad (25)$$

2.1.3 Beam Transfer Function

Let us now introduce model damping through the parameter ζ in the form

$$\ddot{q}_i(t) + 2\zeta\omega_i(\kappa)\dot{q}_i(t) + \omega_i(\kappa)^2 q_i(t) = \frac{2F(t)}{i\pi}\sqrt{\frac{2}{L\rho}}, \qquad i = 1, 3, \ldots, \infty, \tag{26}$$

$$\ddot{q}_i(t) + 2\zeta\omega_i(\kappa)\dot{q}_i(t) + \omega_i(\kappa)^2 q_i(t) = 0, \qquad i = 2, 4, \ldots, \infty \tag{27}$$

and define an output as the beam position at $r \in [0, L]$ computed as

$$Y(t) = \sum_{i=1} \psi_i(r)\, q_i(t). \tag{28}$$

Using Laplace transform methods we can compute the transfer function from F to Y, which takes the form

$$H(s) = \frac{2}{\pi}\sqrt{\frac{2}{L\rho}} \sum_{i=1}^{\infty} \frac{\psi_{2i+1}(r)}{(2i+1)\,(s^2 + 2\,\zeta\,\omega_{2i+1}(\kappa)s + \omega_{2i+1}(\kappa)^2)} \tag{29}$$

We will use this transfer function to evaluate the response of the beam with respect to the parameters of the beam as well as the number and stiffness of the springs.

2.2 Data for Colossus

We use glass with

$$E = 50 \times 10^9\,\mathrm{Pa}, \qquad \rho = 2.5\,\mathrm{g/cm^3} = 2500\,\mathrm{Kg/m^3}. \tag{30}$$

We assume a uniform rectangular cross-section with width b and thickness h so that

$$I = \frac{b\,h^3}{12} \tag{31}$$

In order to make it behave close to the Colossus mirror we set

$$L = 8\,\mathrm{m}, \qquad h = 2.5\,\mathrm{cm}. \tag{32}$$

The total weight of the beam is therefore

$$m = \rho\, L h b \tag{33}$$

We set b so that actuators spaced by 20 cm would support about 4 Kg, i.e. 40 N. In 8 m span we have therefore $8/0.2 - 1 = 39$ actuators and two supports. With each support handling half of the

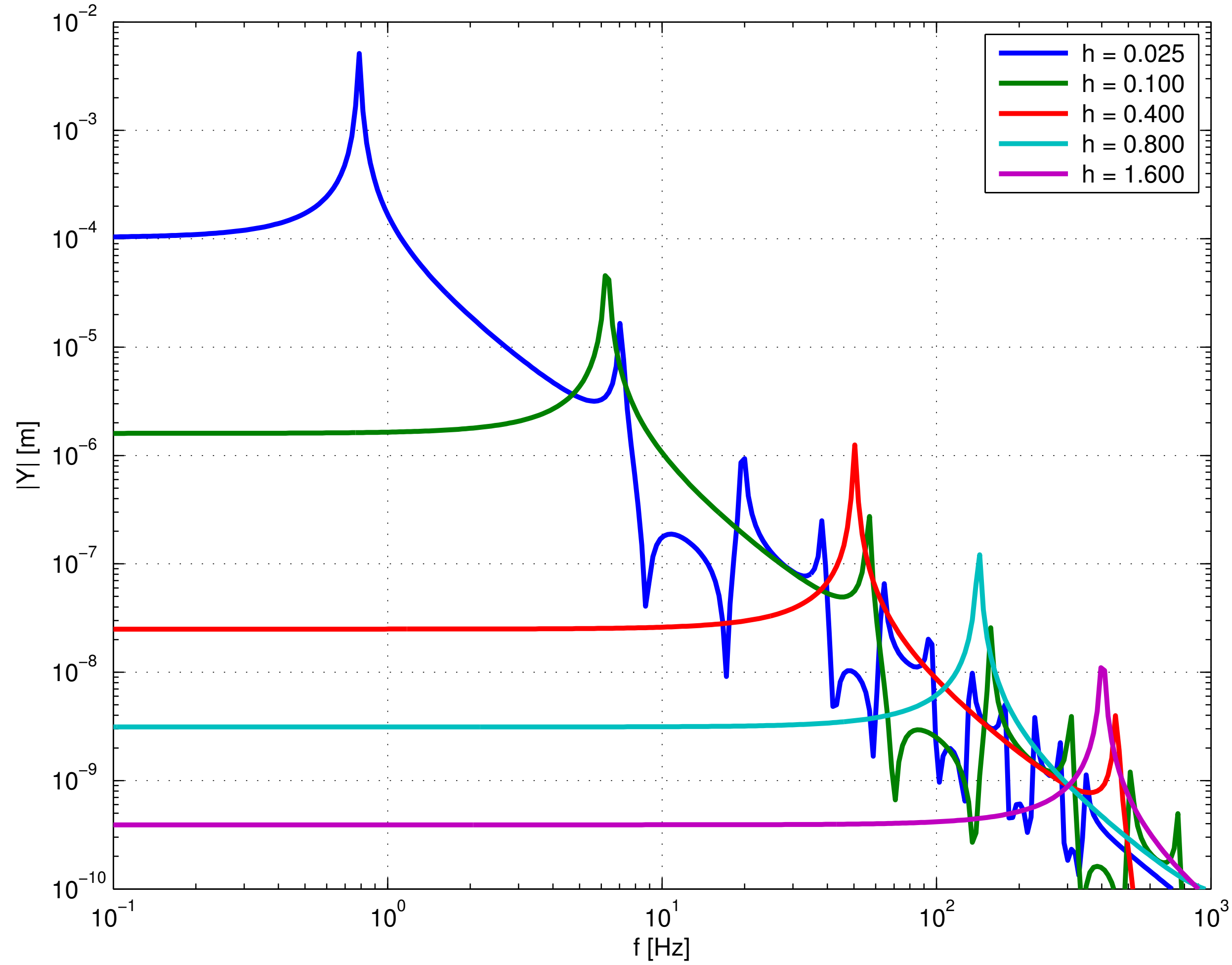


Figure 3: Frequency response of the unsupported mirror for various thicknesses

load of one actuator we should have

$$\rho L h b/40 = 4 \qquad \Longrightarrow \qquad b = \frac{164}{\rho L h} = 0.32\,\mathrm{m} \tag{34}$$

2.2.1 Frequency Response of the Free Mirror

In Fig. 3 we plotted the frequency response of the mirror measured from a uniform time-varying force input to a displacement at the center of the mirror ($r = 4$). The values can be thought as displacements in response to 1 N forces applied to the mirror. Note that for sub-nanometer performance a thickness of $h = 1.6$ m is required in the case of a free unsupported mirror.

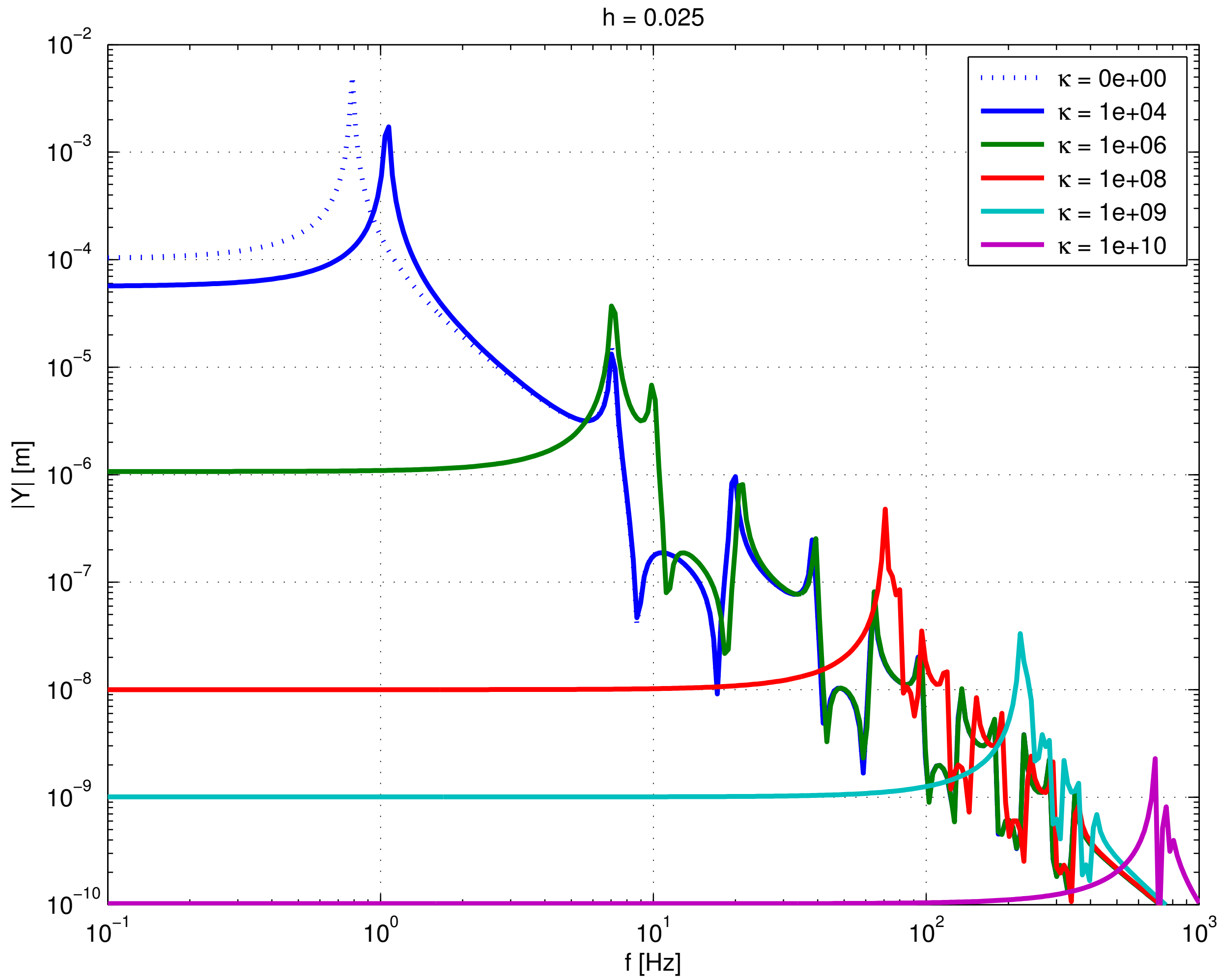


Figure 4: Frequency response of the mirror supported by springs with thickness $h = 2.5$ cm and various spring stiffnesses

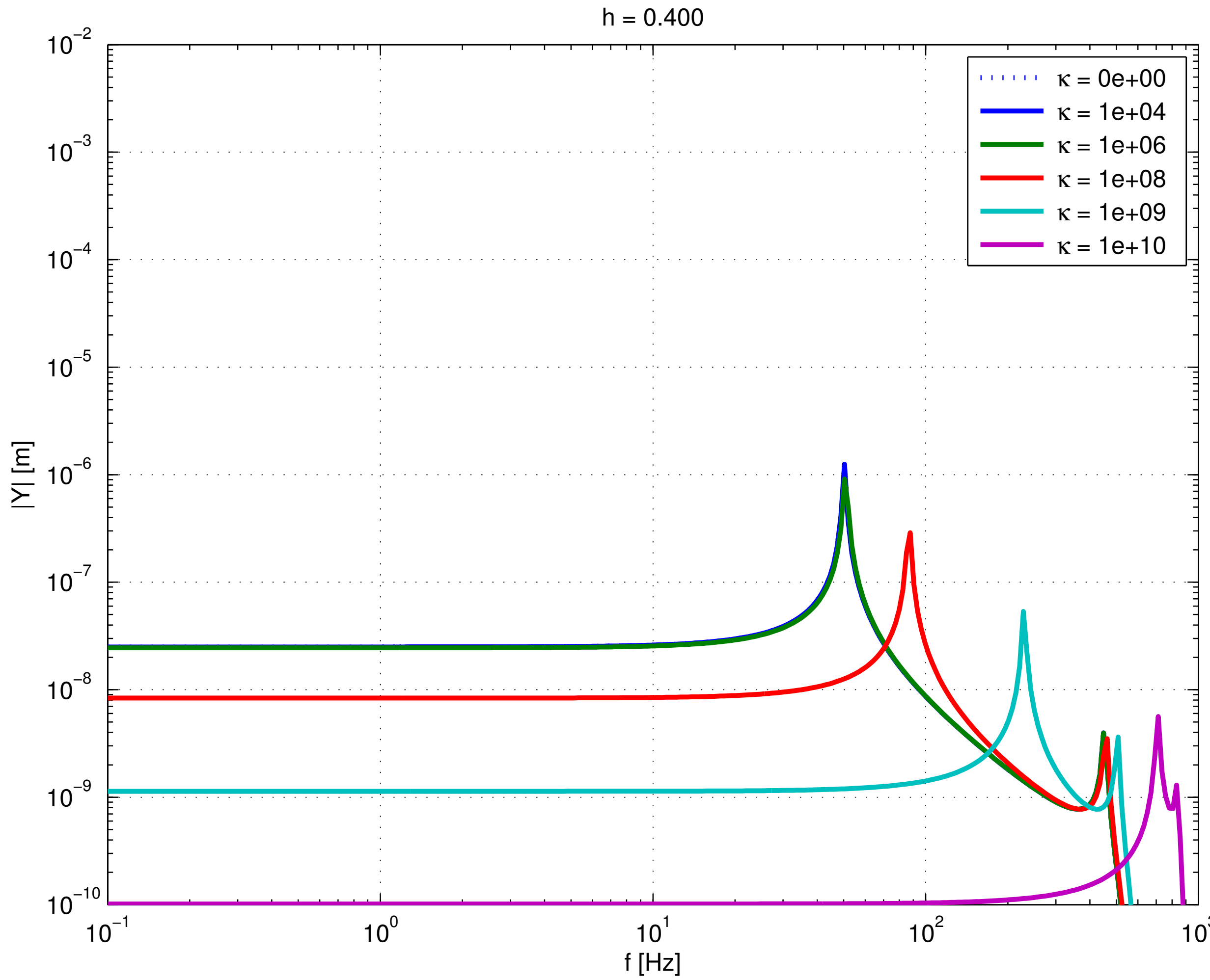


Figure 5: Frequency response of the mirror supported by springs with thickness $h = 40$ cm and various spring stiffnesses

2.2.2 Frequency Response of the Mirror with Spring Support

We now turn to the case when springs are added to support the mirror. In Fig. 4 we plot the frequency response with respect to the stiffness parameter

$$\kappa = (1 + n)\, k$$

and a given mirror thickness $h = 2.5$ cm. For each κ, one has a choice of realizing the required stiffness by increasing the number of springs, n, or increasing the stiffness of the each spring, k. Note that sub-nanometer performance is only possible with very high stiffnesses. For example, 40 steel solid rods of length 10 cm equally spaced by 20 cm would have to have a diameter of 4 cm for $\kappa \approx 10^9$. In practice a very solid foundation (backup structure) would be required to realize such stiffness levels.

Interestingly, increasing the thickness of the glass would have little effect on the mirror system as most of the stiffness is coming from the springs and not the mirror, unless we are willing to use a 2 m thick piece of glass. This is illustrated in Fig. 5, as we plot the frequency response for a thickness $h = 40$ cm.

2.2.3 Covariance Analysis of the Vibration of a One Mode Model of the Beam

One way to understand vibration issues in the presence of stochastic wind disturbances or sensor/actuator noise, is to do a covariance analysis. Here we show a simplified covariance analysis for a single (first) mode of the beam, with wind disturbances characterized as uniformly distributed zero mean white noise $w(t)$ having variance $W > 0$.

We define the variance of the mirror displacement error at the center of the beam, $r = 4$, as $Q = E(Y^2)$. The beam dynamics simplified to contain only the first mode are given by

$$\ddot{q}(t) + 2\,\zeta\,\omega_1(\kappa)\dot{q}(t) + \omega_1(\kappa)^2 q(t), = \frac{2}{\pi}\sqrt{\frac{2}{L\rho}}w(t) \tag{35}$$

$$Y(t) = \sqrt{\frac{2}{L\rho}}q(t). \tag{36}$$

The ratio between the variance of the wind, W, and the variance of the beam displacement, Q, in this simplified model is then given by

$$\frac{W}{Q} = \frac{1}{4}L^2\pi^2\,\zeta\rho^2\omega_1(\kappa)^3 \tag{37}$$

From the above expression it is possible to extract how much wind variance is tolerated for a given κ. Assuming again that $\zeta = 0.01$ we let the standard deviation of the displacement be $\sqrt{Q} = 10$ nm and compute the corresponding standard deviation of the wind, $\sqrt{W}$, that can be tolerated by the mirror. This plot shown in Fig. 6.

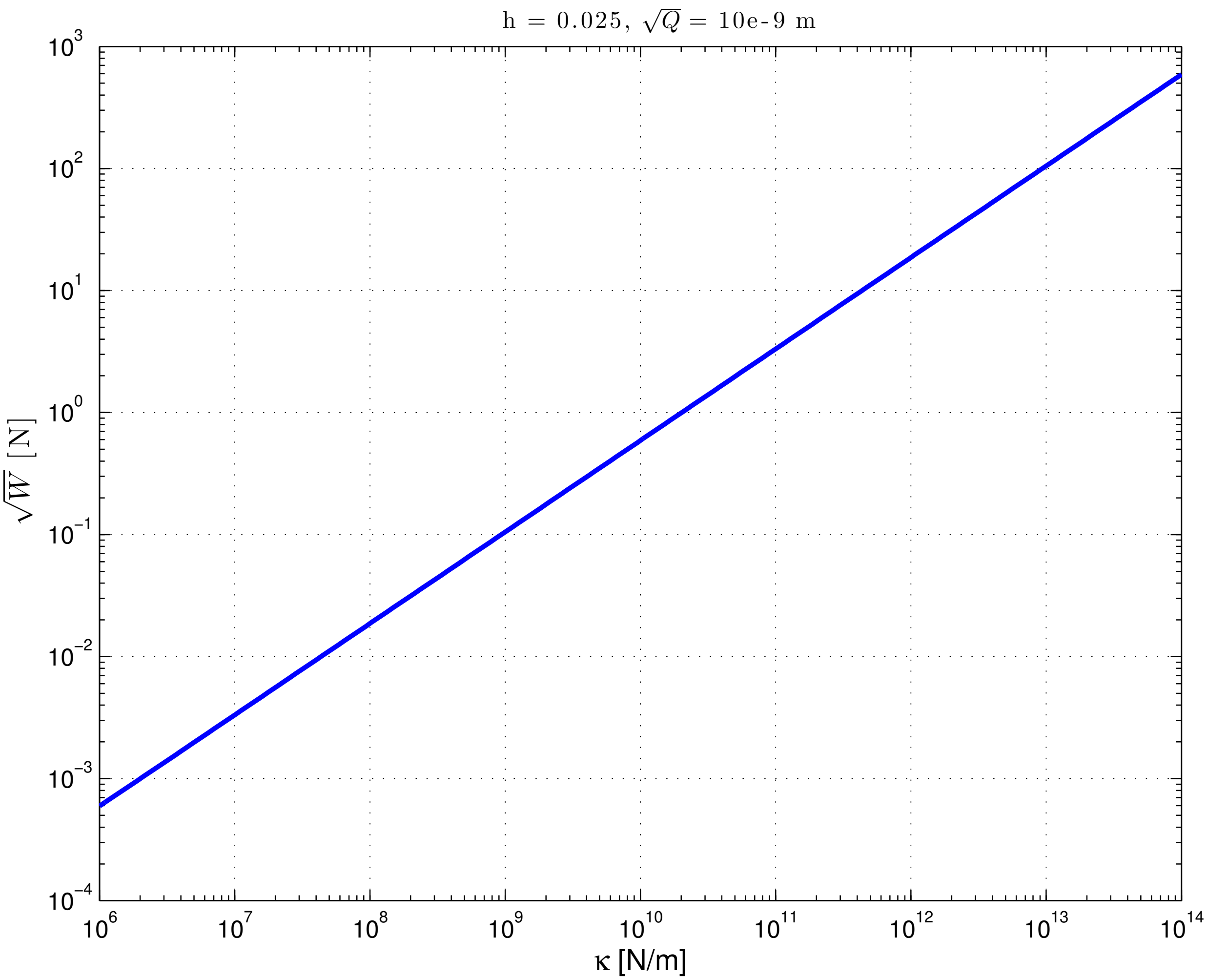


Figure 6: Maximum tolerated wind standard deviation as a function of the stiffness parameter κ for the standard deviation of the mirror displacement to be less than 10 nm.